Projective Geometry

After [N. J. Wildberger's YouTube lecture series
http://www.youtube.com/watch?v=NYK0GBQVngs et al

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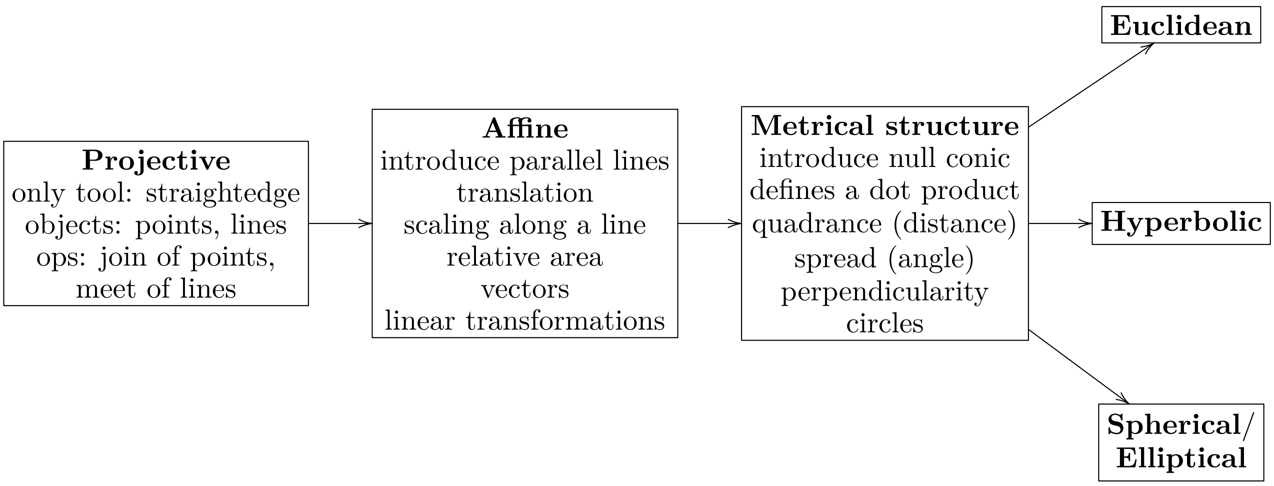
Projective geometry is the geometry involving only a straightedge. It is the fundament for affine, Euclidean, spherical, hyperbolic, and other geometries.

Classification of geometries



Notation

Points: projective points have lower-case variables (a, p); affine (Euclidean) points have upper-case variables (A, P); the product of two points ab is the line that goes through a and b

Lines: projective lines have upper-case variables (L, M); affine lines have lower-case variables (l, m); the product of two lines AB is their intersection, i. e. a point

Important Theorems

Pappus' Theorem: If a_1, a_2, a_3 are collinear and b_1, b_2, b_3 are collinear:
$c_1 := (a_2b_3)(a_3b_2)$, $c_2 := (a_1b_3)(a_3b_1)$, $c_3 := (a_2b_3)(a_3b_2)$ are collinear.

Pascal's Theorem: If $a_1, a_2, a_3, b_1, b_2, b_3$ are points on a conic:
$c_1 := (a_2b_3)(a_3b_2)$, $c_2 := (a_1b_3)(a_3b_1)$, $c_3 := (a_2b_3)(a_3b_2)$ are collinear.

Desargues Theorem: If two triangles $\overline{a_1a_2a_3}$ and $\overline{b_1b_2b_3}$ are perspective from a point p, i. e. a_1b_1, a_2b_2, a_3b_3 are concurrent; then they are perspective from a line L, where $(a_1a_2)(b_1b_2)$, $(a_2a_3)(b_2b_3)$, $(a_3a_1)(b_3b_1)$ are collinear).

Perspectivity

In arts, photography, etc., one often has a horizon and perspective. In these projections, these rules hold:

- Lines are projected to lines.
- All parallel lines meet at a point on the horizon ("point at infinity"). There is one point on the horizon for each family of parallel lines.
- Conics are projected to conics. E. g. a parabola or a hyperbola will look like an ellipse. All conics will look elliptic as they originate from slicing a cone.

Projective homogeneous coordinates

In affine geometry, lines from a 3D object through the projective plane are parallel, and projections maintain linear spacing. In projective geometry, they meet in two points ("observer" or "horizon"), and projections distort linear spacing.

In **one-dimensional geometry**, the projective line $\mathbb{P}^1$ is the space of one-dimensional subspaces of the two-dimensional $\mathbb{V}^2$, i. e. a line through $[0, 0]$. This is specified by a proportion $[x : y]$ (**projective point**); canonically, either $[x : 1]$, or $[x : 0]$ for the special case of the subspace being the x axis (representing "point at ∞")

In **two-dimensional geometry**, the projective plane $\mathbb{P}^2$ is described with a three-dimensional vector space $\mathbb{V}^3$, **projective points** $a = [x : y : z]$ (lines through the origin) and **projective lines** $A = (l : m : n) \Leftrightarrow lx + my + nz = 0$ (planes through the origin).

In particular, we introduce a **viewing plane** $y_{\mathrm{z=1}}$. Then any projective point $[x : y : z], z \neq 0$ meets the viewing plane at $[\frac{x}{z} : \frac{y}{z} : 1]$. Any projective point $[x : y : 0]$ corresponds to the two-dimensional equivalent $[x : y]$, representing the "point at ∞ in the direction $[x : y]$". So the projective plane $\mathbb{P}^2$ corresponds to the affine plane $\mathbb{A}^2$ plus the projective line $\mathbb{P}^1$ for the points at infinity.

Any projective line $(l : m : n), l, m \neq 0$ meets the viewing plane at the line $lx + my = n$.

Coordinates in the viewing plane are called $X := \frac{x}{z}$ and $Y := \frac{y}{z}$.

This introduces a duality between lines and points: Any theorem has a dual counterpart where lines and points, meets and joins etc. are interchanged.

Calculations with homogenous coordinates

Join of points: If $a = [x_1 : y_1 : z_1]$ and $b = [x_2 : y_2 : z_2]$, then

$$ab = (y_1 z_2 - y_2 z_1 : z_1 x_2 - z_2 x_1 : x_1 y_2 - x_2 y_1)$$

Meet of lines: If $L = (l_1 : m_1 : n_1)$ and $M = (l_2 : m_2 : n_2)$, then

$$ab = [m_1 n_2 - m_2 n_1 : n_1 l_2 - n_2 l_1 : l_1 m_2 - l_2 m_1]$$

Incidence: The point $[x : y : z]$ and the line $(l : m : n)$ are incident $\Leftrightarrow lx + my + nz = 0$

Linear transformation: A linear transformation $(x, y) \to (x', y') = (x\ y) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is represented homogeneously with $[x : y : z] \to [x' : y' : z'] = [x\ y\ z] \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Affine transformation: A translation (non-linear) $(x, y) \to (x+a, y+b)$ is represented homogeneously with $[x : y : z] \to [x' : y' : z'] = [x\ y\ z] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix} = [x + az : y + bz : z]$

Metrical structure

Add a "null conic" $\mathcal{C}$ to the geometry by defining a (homogeneous) symmetric matrix $A = \begin{bmatrix} a & d & f \\ d & b & g \\ f & g & c \end{bmatrix}$

For any point $p = [x : y : z]$, setting $pAp^T = 0$ expands to the general equation of a homogeneous conic: $ax^2 + by^2 + cz^2 + 2dxy + 2fxz + 2gyz = 0$

When dividing by z^2 we get the affine coordinates: $aX^2 + bY^2 + 2dXY + 2fX + 2gY + c = 0$

This matrix A defines a symmetric bilinear form (**dot/inner product**) on $\mathbb{V}^3$: $v \cdot w := vAw^T$

The usual **Euclidean** one is $A_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ but there are other interesting ones too, like the **relativistic** (Lorentz, Minkowski, Einstein) structure $A_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

This defines the notion of **perpendicularity**: $p \perp q \Leftrightarrow p \cdot q = pAq^T = 0$. This is well-defined projectively and in $\mathbb{P}^2$.

Polarity

$aAb^T = 0$ can be interpreted as $a(Ab^T) = 0$ and since $L := Ab^T$ defines a line, as $aL = 0$. I. e. the dot product defines a duality between points and lines, called **polarity**.

Pole: The pole a of a line L is the inversion point of L's closest point to $\mathcal{C}$. $\mathbf{a} = \mathbf{L}^\perp$

Polar: The line L is the polar of the pole point a. $\mathbf{L} = \mathbf{a}^\perp$

Construction of polar of a:

- Find "null points" $\alpha, \beta, \gamma, \delta$ on $\mathcal{C}$ so that $(\alpha\beta)(\gamma\delta) = a$ (i. e. select two secants of $\mathcal{C}$ which pass through a)
- Define $b := (\alpha\delta)(\beta\gamma)$ and $c := (\alpha\gamma)(\beta\delta)$, i. e. the two intersections of the other diagonals of the quadrangle $\alpha\beta\gamma\delta$
- Then $A = a^\perp = bc$. Also, $b^\perp = ac$ and $c^\perp = ab$ (three-fold symmetry of polars and poles produced by the three intersections of diagonals)
- **Polar Independence Theorem:** The polar $A = a^\perp$ does not depend on the choice of lines through a, i. e. the choice of $\alpha, \beta, \gamma, \delta$.

Construction of pole of A: Choose any two points $b, c \in A$. Then $a = (b^\perp c^\perp)$.

Construction of polar of null point γ:

- Chose any two secants A, B that pass through γ
- Construct their poles $A^\perp$, $B^\perp$
- Polar $\Gamma = \gamma^\perp$ is the tangent $(A^\perp B^\perp)$

Polar Duality Theorem: For any two points a and b: $a \in b^\perp \Leftrightarrow b \in a^\perp$ (as A is symmetric)