Universal Hyperbolic Geometry

Summary of N. J. Wildberger's online lecture series

http://www.youtube.com/watch?v=EvP8VtyhzXs

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1 Notation

Conics: intersections of a (double-sided) cone with a plane: point, circle, elipse, parabola, hyperbola (consists of two separate parts)

Null conic: Single fixed given circle or other conic; symbol $\mathcal{C}$

Points: points have lower-case variables (a, p); points on $\mathcal{C}$ have lower-case greek variables (α, η); the product of two points ab is the line that goes through a and b
algebraic coordinate notation: $a = [x]$ (one-dimensional), $a = [x, y]$ (two-dimensional)

Lines: lines have upper-case variables (A, B); tangents on $\mathcal{C}$ have upper-case greek variables (Γ); the product of two lines AB is their intersection, i. e. a point
algebraic coordinate notation: $A := (a{:}b{:}c)$ representing $ax + by = c$

2 Polarity

http://www.youtube.com/watch?v=AjVM5Q-pvjw

Pole: The pole a of a line A is the inversion point of A's closest point to $\mathcal{C}$. $\mathbf{a} = \mathbf{A}^\perp$

Polar: The line A is the polar of the pole point a. $\mathbf{A} = \mathbf{a}^\perp$

Construction of polar of a:

- Find null points $\alpha, \beta, \gamma, \delta$ on $\mathcal{C}$ so that $(\alpha\beta)(\gamma\delta) = a$ (i. e. select two secants of $\mathcal{C}$ which pass through a)
- Define $b := (\alpha\delta)(\beta\gamma)$ and $c := (\alpha\gamma)(\beta\delta)$, i. e. the two intersections of the other diagonals of the quadrangle $\alpha\beta\gamma\delta$
- Then $A = a^\perp = bc$. Also, $b^\perp = ac$ and $c^\perp = ab$ (three-fold symmetry of polars and poles produced by the three intersections of diagonals)

Construction of pole of A: Choose any two points $b, c \in A$. Then $a = (b^\perp c^\perp)$.

Polar Independence Theorem: The polar $A = a^\perp$ does not depend on the choice of lines through a, i. e. the choice of $\alpha, \beta, \gamma, \delta$.

Polar Duality Theorem: For any two points a and b: $a \in b^\perp \Leftrightarrow b \in a^\perp$

Construction of polar of null point γ:

- Chose any two secants A, B that pass through γ
- Construct their poles $A^\perp$, $B^\perp$
- Polar $\Gamma = \gamma^\perp$ is the tangent $(A^\perp B^\perp)$

3 Harmonic conjugates

http://www.youtube.com/watch?v=t7oXlrcPBb4

Four collinear points a, b, c, d are a **harmonic range** if a and c are **harmonic conjugates** to b, d, i. e. if they divide $\overline{bc}$ internally and externally by the same ratio:

$$\frac{\vec{ab}}{\vec{ad}} = -\frac{\vec{cb}}{\vec{cd}}$$

In that case, b and d are then harmonic conjugates to a and c:

$$\frac{\vec{ba}}{\vec{bc}} = -\frac{\vec{da}}{\vec{dc}}$$

Note: $\vec{ab}$ measures displacements on a linear scale, not distances. (affine geometry only, no units)

Harmonic Ranges Theorem: The image of a harmonic range under a projection from a point onto another line is another harmonic range.
$\rightarrow$ harmonic ranges are not dependent on the choice of a scale (affine geometry), but are really part of projective geometry

If a, b, c, d are a harmonic range, and p a point not on the line $abcd$, then the four lines ap, bp, cp, dp are a **harmonic pencil**. By the previous theorem, the intersections of any line through these four lines are harmonic ranges.

Harmonic Pole/polar Theorem: For a point a and any secant through a that meets $\mathcal{C}$ at two points β, γ there is a point $c = (\beta\gamma)a^\perp$. The points a, β, c, δ are a harmonic range.

Harmonic Bisectors Theorem: If C, D are the two bisectors of two non-parallel lines A, B, then A, C, B, D is a harmonic pencil.

Harmonic Vectors Theorem: If $\vec{a}, \vec{b}$ are linearly independent vectors, then the lines spanned by $\vec{a}, \vec{b}$ are harmonic conjugates to the lines spanned by $\vec{a} + \vec{b}, \vec{a} - \vec{b}$

Harmonic Quadrangle Theorem: If $\overline{pqrs}$ is a quadrangle, find the two intersections $a = (ps)(qr)$ and $c = (pq)(rs)$ and then $b = (pr)(ac)$, $d = (sq)(ac)$. Then a, b, c, d are a harmonic range.

4 Cross ratios

http://www.youtube.com/watch?v=JJbh0iJ1Agc

Cross ratio of four collinear points a, b, c, d:

$$R(a, b : c, d) := \frac{\vec{ac}}{\vec{ad}} \Big/ \frac{\vec{bc}}{\vec{bd}} \quad = \quad \frac{a-c}{a-d} \Big/ \frac{b-c}{b-d}$$

a, b, c, d are a harmonic range if $R(a, b : c, d) = -1$.

Cross-ratio Transformation Theorem: If $R(a, b : c, d) = \lambda$ then

$$R(b, a : c, d) = R(a, b : d, c) = \frac{1}{\lambda}$$

and

$$R(a, c : b, d) = R(d, b : c, a) = 1 - \lambda$$

Four points determine 24 possible cross ratios, but only 6 will generally be different (permutations of λ, $\frac{1}{\lambda}$, $1-\lambda$, $\frac{1}{1-\lambda}$, $\frac{\lambda-1}{\lambda}$, $\frac{\lambda}{\lambda-1}$).

Cross-ratio Theorem: The cross ratio is invariant under projection from a point p to another line L: With $a' := (pa)L$, $b' := (pb)L$, $c' := (pc)L$, $d' := (pd)L$:

$$R(a, b : c, d) = R(a', b' : c', d')$$

Therefore the cross-ratio can be transferred to lines:

$$R(pa, pb, pc, pd) := R(a, b, c, d)$$

Chasles Theorem: If $\alpha, \beta, \gamma, \delta$ are fixed points on a null conic $\mathcal{C}$ and η a fifth point on $\mathcal{C}$, then $R(\alpha\eta, \beta\eta : \gamma\eta, \delta\eta)$ is independent of the choice of η.

5 Introduction to hyperbolic geometry

http://www.youtube.com/watch?v=UXQas-B5ObQ

5.1 Definitions

Hyperbolic geometry : Geometry on a projective plane and a single fixed given circle $\mathcal{C}$; only tool is a straightedge

Duality: Terminology of pole and polar get replaced by simply **duality**: the dual of a point a is the line $a^\perp$, and vice versa; points and lines are completely dual concepts

Line perpendicularity: $A \perp B \Leftrightarrow B^\perp \in A \Leftrightarrow A^\perp \in B$ (A is p. to B if A passes through the dual of B)

Point perpendicularity: $a \perp b \Leftrightarrow a \in b^\perp \Leftrightarrow b \in a^\perp$ (a is p. to b if a lies on the dual of b)

Quadrance between points: $q(a_1, a_2) := R(a_1, b_2 : a_2, b_1)$ with $b_1 := (a_1a_2)a_1^\perp$ and $b_2 := (a_1a_2)a_2^\perp$

Spread between lines: $S(A_1, A_2) := R(A_1, B_2 : A_2, B_1)$ with $B_1 := (A_1A_2)A_1^\perp$ and $B_2 := (A_1A_2)A_2^\perp$

5.2 Basic theorems

Quadrance–Spread duality: $q(a_1, a_2) = S(a_1^\perp, a_2^\perp)$

Pythagoras: If $a_1a_3 \perp a_2a_3$, and $q_1 = q(a_2, a_3)$, $q_2 = q(a_1, a_3)$, $q_3 = q(a_1, a_2)$: $q_3 = q_1 + q_2 - q_1q_2$

If $A_1A_3 \perp A_2A_3$, and $S_1 = S(A_2, A_3)$, $S_2 = S(A_1, A_3)$, $S_3 = S(A_1, A_2)$: $S_3 = S_1 + S_2 - S_1S_2$

Triple Spread/Quadrance:

If a_1, a_2, a_3 are collinear: $(q_1 + q_2 + q_3)^2 = 2(q_1^2 + q_2^2 + q_3^2) + 4q_1q_2q_3$

If A_1, A_2, A_3 are concurrent: $(S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3$

Spread Law: For a triangle: $\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}$

Cross law: $(q_1q_2S_3 - (q_1 + q_2 + q_3) + 2)^2 = 4(1 - q_1)(1 - q_2)(1 - q_3)$

Relation to Beltrami-Klein model:

For points a_1, a_2 inside $\mathcal{C}$: $q(a_1, a_2) = -\sinh^2 d(a_1, a_2)$

For lines A_1, A_2 inside $\mathcal{C}$: $S(A_1, A_2) = \sin^2 \angle(A_1, A_2)$

6 Calculations with Cartesian coordinates

http://www.youtube.com/watch?v=YDGUnGGkaTs, http://www.youtube.com/watch?v=XomxP2pxYnw

Point/line duality: $a = [x_0, y_0] \quad \Leftrightarrow \quad a^\perp = (x_0 : y_0 : 1)$

Point on null circle: Parameterized by $t \in \mathbb{Q}$: $e(t) := \left[\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right]$

This reaches any point except $[-1, 0]$, which corresponds to $e(\infty)$

Line through points: $[x_1, y_1][x_2, y_2] = (y_1 - y_2 : x_2 - x_1 : x_2y_1 - x_1y_2)$

Line through null points: $e(t_1)e(t_2) = (1 - t_1t_2 : t_1 + t_2 : 1 + t_1t_2)$

Line relation to $\mathcal{C}$: The line $(a : b : c)$

- is tangent to $\mathcal{C} \Leftrightarrow a^2 + b^2 = c^2$
- meets $\mathcal{C}$ at two points $\Leftrightarrow a^2 + b^2 - c^2$ is a square
- does not meet $\mathcal{C} \Leftrightarrow a^2 + b^2 - c^2$ is a non-square (negative or no rational root)

Quadrance: If $a_1 = [x_1, y_1]$ and $a_2 = [x_2, y_2]$, then $q(a_1, a_2) = 1 - \frac{(x_1x_2+y_1y_2-1)^2}{(x_1^2+y_1^2-1)(x_2^2+y_2^2-1)}$

Spread: If $L_1 = (l_1 : m_1 : n_1)$ and $L_2 = (l_2 : m_2 : n_2)$, then $S(L_1, L_2) = 1 - \frac{(l_1l_2+m_1m_2-n_1n_2)^2}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)}$

Point perpendicularity: $[x_1, y_1] \perp [x_2, y_2] \quad \Leftrightarrow \quad x_1x_2 + y_1y_2 - 1 = 0$

Line perpendicularity: $(l_1 : m_1 : n_1) \perp (l_2 : m_2 : n_2) \quad \Leftrightarrow \quad l_1l_2 + m_1m_2 - n_1n_2 = 0$

7 Projective homogeneous coordinates

http://www.youtube.com/watch?v=gzalbNLcwGI, http://www.youtube.com/watch?v=KhxVy75NetE

In affine geometry, lines from a 3D object through the projective plane are parallel, and projections maintain linear spacing. In projective geometry, they meet in two points ("observer" or "horizon"), and projections distort linear spacing.

In **one-dimensional geometry**, the projective vector space is an one-dimensional subspace of two-dimensional $\mathbb{V}^2$, i. e. a line through $[0, 0]$). This is specified by a proportion $[x : y]$ (**projective point**); canonically, either $[x : 1]$, or $[x : 0]$ for the special case of the subspace being the x axis (representing "point at ∞")

In **two-dimensional geometry**, the projective plane $\mathbb{P}^2$ is described with a three-dimensional vector space $\mathbb{V}^3$, **projective points** $a = [x : y : z]$ (lines through the origin) and **projective lines** $A = (l : m : n) \Leftrightarrow lx + my - nz = 0$ (planes through the origin).

In particular, we introduce a **viewing plane** $y_{z=1}$. Then any projective point $[x : y : z], z \neq 0$ meets the viewing plane at $[\frac{x}{z} : \frac{y}{z} : 1]$. Any projective point $[x : y : 0]$ corresponds to the two-dimensional equivalent $[x : y]$, representing the "point at ∞ in the direction $[x : y]$". So the projective plane $\mathbb{P}^2$ corresponds to the affine plane $\mathbb{A}^2$ plus the projective line $\mathbb{P}^1$ for the points at infinity.

Any projective line $(l : m : n), l, m \neq 0$ meets the viewing plane at the line $lx + my = n$ (which is $(l : m : n)$ in the viewing plane).

Coordinates in the viewing plane are called $X := \frac{x}{z}$ and $Y := \frac{y}{z}$.

The unit circle $X^2 + Y^2 = 1$ in the viewing plane corresponds to the cone $(\frac{x}{z})^2 + (\frac{y}{z})^2 = 1 \Rightarrow x^2 + y^2 - z^2 = 0$ in $\mathbb{A}^3$.

8 Calculations with homogenous coordinates

http://www.youtube.com/watch?v=tk58sBLWzHk, http://www.youtube.com/watch?v=N2TObg_DJLQ, http://www.youtube.com/watch?v=PSFr6_EhchI

(Hyperbolic) Point: a proportion $a := [x{:}y{:}z]$ with $x, y, z \in \mathbb{Q}$ and not all zero

(Hyperbolic) Line: a proportion $L := (l{:}m{:}n)$ with $l, m, n \in \mathbb{Q}$ and not all zero

Duality: $a = [x{:}y{:}z] \Leftrightarrow a^\perp = (x{:}y{:}z) \qquad L = (l{:}m{:}n) \Leftrightarrow L^\perp = [l{:}m{:}n]$

Line/Point Incidence: $[x{:}y{:}z]$ lies on $(l{:}m{:}n) \Leftrightarrow (l{:}m{:}n)$ goes through $[x{:}y{:}z] \Leftrightarrow lx + my - nz = 0$

Line through two points: $[x_1{:}y_1{:}z_1][x_2{:}y_2{:}z_2] = (y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_2y_1 - x_1y_2)$

Point on two lines: $(l_1{:}m_1{:}n_1)(l_2{:}m_2{:}n_2) = [m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : l_2m_1 - l_1m_2]$

Quadrance: If $a_1 = [x_1{:}y_1{:}z_1]$ and $a_2 = [x_2{:}y_2{:}z_2]$, then $q(a_1, a_2) = 1 - \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}$

$$a_1 \perp a_2 \;\Leftrightarrow\; q(a_1, a_2) = 1$$

Spread: If $L_1 = (l_1 : m_1 : n_1)$ and $L_2 = (l_2 : m_2 : n_2)$, then $S(L_1, L_2) = 1 - \frac{(l_1l_2+m_1m_2-n_1n_2)^2}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)}$

$$L_1 \perp L_2 \;\Leftrightarrow\; S(L_1, L_2) = 1$$

9 Triangle geometry

http://www.youtube.com/watch?v=PbA4Js3qKOQ, http://www.youtube.com/watch?v=Drs8hUPzRP0

Side: A side $\overline{a_1a_2}$ is a set of two points $\{a_1, a_2\}$.

Vertex: A vertex $\overline{L_1L_2}$ is a set of two lines $\{L_1, L_2\}$.

Couple: A couple $\overline{aL}$ is a set consisting of a point and a line: $\{a, L\}$. A couple is **dual** if $a = L^\perp$.

Triangle: A triangle $\overline{a_1a_2a_3}$ is a set of three non-collinear points $\{a_1, a_2, a_3\}$. A triangle is **dual** if one of its points is dual to its opposite side.

Dual trilateral: $\overline{a_1a_2a_3}^\perp := \overline{a_1^\perp a_2^\perp a_3^\perp}$ (similar for dual triangle)

Trilateral: A trilateral $\overline{L_1L_2L_3}$ is a set of three non-concurrent lines $\{L_1, L_2, L_3\}$.

Altitude Line Theorem: For any non-dual couple $\overline{aL}$ there is an unique line N passing through a and perpendicular to L, called **altitude**. $N = aL^\perp$

Altitude Point Theorem: For any non-dual couple $\overline{aL}$ there is an unique point n which lies on L and is perpendicular to a, called **altitude point**. $n = a^\perp L = N^\perp$

Triangle Altitudes: In a triangle $\overline{abc}$, altitudes are determined in the usual way: e. g. the altitude of a goes through a and is perpendicular to bc. Thus the altitude of a is $a(bc)^\perp$.

Orthocenter: Point h where the three altitudes meet; always exists

Ortholine: Line H on which the three altitude points are collinear; $H = h^\perp$

Desargues Theorem: In the projective plane, if two triangles $\overline{a_1a_2a_3}$ and $\overline{b_1b_2b_3}$ are perspective from a point p (a_1b_1, a_2b_2, a_3b_3 are concurrent) then they are perspective from a line L (where $(a_1a_2)(b_1b_2)$, $(a_2a_3)(b_2b_3)$, $(a_3a_1)(b_3b_1)$ are collinear). **Desargues polarity:** $L = \hat{p}$

Definitions relative to fixed triangle: **Orthic axis** $S = \hat{h}$, **Orthostar** $s = S^\perp$, **Ortho-axis** $A = hs$, **Ortho-axis point** $a = A^\perp$ (lies on ortholine)

Orthic triangle: Triangle $\overline{b_1b_2b_3}$ of the base points of altitudes of triangle $\overline{a_1a_2a_3}$; $b_1 = (a_1h)(a_2a_3)$

Base center: A triangle and its dual orthic triangle are perspective from some point, called the **base center**, i. e. the intersection of all lines through a triangle point with its corresponding dual orthic triangle point. It lies on the ortho-axis $A = hs$.

Base triple orthocenter theorem: Suppose that the triangle $\overline{a_1a_2a_3}$ has the orthic triangle $\overline{b_1b_2b_3}$. Suppose that h_1, h_2, h_3 are the respective orthocenters of $\overline{a_1b_2b_3}$, $\overline{a_2b_1b_3}$, and $\overline{a_3b_1b_2}$. Then the orthocenter of $\overline{h_1h_2h_3}$ is the base center b of $\overline{a_1a_2a_3}$. Also, b is the center of perspectivity between $\overline{a_1a_2a_3}$ and $\overline{h_1h_2h_3}$.

Quadrea: $A(\overline{a_1a_2a_3}) = q_1q_2S_3 = q_2q_3S_1 = q_1q_3S_2 =$

$$-\frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)}$$

(most important triangle invariant)

Equilateral Triangle Theorem: If a triangle has three equal non-zero quadrances q, then it also has three equal spreads S, and $(1 - Sq)^2 = 4(1 - S)(1 - q)$.

Thales Theorem: Immediately following from the spread law: in a right triangle with $S_3 = 1$: $S_1 = \frac{q_1}{q_3}$ and $S_2 = \frac{q_2}{q_3}$.

Corollary: If h_3 is the quadrance of the altitude of a_3 to a_1a_2, then the quadrea $A = h_3q_3$. (Similar for the other two altitudes)

Napier's rules: In a right triangle ($S_3 = 1$), if any two of S_1, S_2, q_1, q_2, q_3 are known, the other three follow from Pythagoras' and Thales' theorems.

10 Null points and null lines

http://www.youtube.com/watch?v=IhEXH5etvog

Null point: A point $a = [x{:}y{:}z]$ is null $\Leftrightarrow$ a is incident with $a^\perp$ $\Leftrightarrow$ $x^2 + y^2 - z^2 = 0$

Null line: A line $L = (l{:}m{:}n)$ is null $\Leftrightarrow$ L is incident with $L^\perp$ $\Leftrightarrow$ $l^2 + m^2 - n^2 = 0$

Null point parameterization: $\alpha = e(t{:}u) := [u^2 - t^2 : 2ut : u^2 + t^2]$

Null line parameterization: $\Phi = E(t{:}u) := (u^2 - t^2 : 2ut : u^2 + t^2)$

Join of null points: $e(t_1{:}u_1)e(t_2{:}u_2) = (u_1u_2 - t_1t_2 : t_1u_2 + t_2u_1 : u_1u_2 + t_1t_2)$

Meet of null lines: $E(t_1{:}u_1)E(t_2{:}u_2) = [u_1u_2 - t_1t_2 : t_1u_2 + t_2u_1 : u_1u_2 + t_1t_2]$

11 Reflections

http://www.youtube.com/watch?v=faPCRHyzPGM, http://www.youtube.com/watch?v=elDCJmDQBfc

Unlike the Euclidean plane, the projective plane is not orientable. A reflection σ_a in a point a is the same as a reflection σ_L in a line if $a = L^\perp$.

Lines are reflected to lines ($M = \sigma_a L$), null points to null points, null lines to null lines.

Construction: To determine the reflection $c = b\sigma_a$ of a point b in a point a:

- Choose a line through b which crosses $\mathcal{C}$ in two points β_1 and β_2. (c does not depend on this choice)
- Determine the two other intersections γ_1 and γ_2 with $\mathcal{C}$ of the two lines $a\beta_1$ and $a\beta_2$.
- Now $c = (ab)(\gamma_1\gamma_2)$.

Reflection matrix: A point $a = [x{:}y{:}z]$ defines a reflection matrix $m_a = \begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix}$

For any m_a: $m_a^2 = \mathbf{1}$, so that $m_a^{-1} = m_a$

For null points $\alpha = e(t : u)$: $m_\alpha = \begin{bmatrix} tu & u^2 \\ -t^2 & -tu \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} t \\ u \end{bmatrix} \begin{bmatrix} t & u \end{bmatrix}$

For any m_α: $m_\alpha^2 = \mathbf{0}$, $\det m_\alpha = 0$

Reflection of null points: With projective linear algebra, a reflection at the point $a = [x{:}y{:}z]$ sends the null point $\alpha_1 = e(t_1{:}u_1)$ to another null point $\alpha_2 = e(t_2{:}u_2)$. Then $[t_2u_2] = [t_1u_1]\, m_a$. Also, $m_{\alpha_2} = m_\alpha m_{\alpha_1} m_\alpha$ (reflection matrix conjugation theorem).

Reflection of an arbitrary point: $c = b\sigma_a \Leftrightarrow m_c = m_a m_b m_a$

Null reflection theorem: If α is a null point, then $b\sigma_\alpha = \alpha$ for any point b. $m_\alpha m_b m_\alpha = m_\alpha$.

Matrix perpendicularity theorem: For any points a, b: $a \perp b \Leftrightarrow \mathrm{tr}(m_a m_b) = 0$

Reflection preserves perpendicularity: For any two points b, c, and a non-null point a:
$b \perp c \Leftrightarrow b\sigma_a \perp c\sigma_a$

Reflection preserves lines: If a is a non-null point, then b, c, d are collinear $\Leftrightarrow b\sigma_a, c\sigma_a, d\sigma_a$ are collinear.

12 Midpoints and bisectors

http://www.youtube.com/watch?v=gYqp_m7at2

Midpoint: The non-null point a is a midpoint of the side $\overline{bc} \Leftrightarrow b\sigma_a = c \Leftrightarrow c\sigma_a = b$
In general there are two different points with that property for $\overline{bc}$, if both points are interior or exterior.

Geometrical construction of midpoints: For points b, c:

- Construct $(bc)^\perp$
- Construct the lines $b(bc)^\perp$ and $c(bc)^\perp$, yielding four null points α, β, γ, δ
- The other two diagonal points of $\overline{\alpha\beta\gamma\delta}$ are the two midpoints of $\overline{bc}$.

Sometimes that constructions does not work because $b(bc)^\perp$ and $c(bc)^\perp$ don't meet $\mathcal{C}$. In that case:

- Construct $b^\perp$ and $c^\perp$, which meet $\mathcal{C}$ in four null points α, β, γ, δ
- The other two diagonal points of $\overline{\alpha\beta\gamma\delta}$ are the two midpoints of $\overline{bc}$.

Bisector: A is a bisector of the vertex $\overline{BC} \Leftrightarrow A^\perp$ is a midpoint of side $\overline{B^\perp C^\perp}$

13 The J function, SL(2) and the Jacobi identity

http://www.youtube.com/watch?v=f68eYuDCsjw

SL(2) : Lie algebra/group of 2x2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $tr(A) = 0$ and a **bracket** operation $[A, B] := AB - BA$ (closed operation as $tr(AB - BA) = tr(AB) - tr(BA) = 0$)

Properties of bracket operation: [] is not associative, and anticommutative ($[A, B] = -[B, A]$)

Jacobi identity: $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$

Projective matrix algebra: If a, b are projective matrices, scaling with non-zero scalars does not change the matrix. $m+n$ is *not* well defined. The product ab and bracket $[a, b] = ab - ba$ are well defined.

Bracket theorem: If a and b are distinct points:$[m_a, m_b] = m_c$ with $c = (ab)^\perp$
The bracket operation gives a multiplication of points and computes their join: $[a, b] \simeq (ab)^\perp$. It is commutative, as the negative of a projective matrix is identical to the matrix.

Meaning of Jacobi identity: For three points a, b, c the term $[[a, b], c]$ is the altitude point (dual of the altitude) of c to ab. As all three altitudes/altitude points sum up to 0, they are linearly dependent and thus concurrent/collinear. $\Rightarrow$ simplified proof that ortholine/orthocenter always exist.

14 Miscellaneous

Pappus' Theorem: If a_1, a_2, a_3 are collinear and b_1, b_2, b_3 are collinear:
$c_1 := (a_2b_3)(a_3b_2)$, $c_2 := (a_1b_3)(a_3b_1)$, $c_3 := (a_2b_3)(a_3b_2)$ are collinear.

Zero Quadrance Theorem: If a_1, a_2 are distinct points, then $q(a_1, a_2) = 0 \Leftrightarrow a_1a_2$ is a null line.

Zero Spread Theorem: If L_1, L_2 are distinct lines, then $S(L_1, L_2) = 0 \Leftrightarrow L_1L_2$ is a null point.

Right Parallax Theorem: If a right triangle $\overline{a_1a_2a_3}$ has spreads $S_1 = 0$ (i. e. a_1 is a null point), $S_2 := S \neq 0$, and $S_3 = 1$, then it will have only one defined quadrance $q = q(a_2, a_3) = \frac{S-1}{S}$.

Isosceles Parallax Theorem: If $\overline{a_1a_2a_3}$ is a non-null isosceles triangle with $S_1 = 0$ (i. e. a_1 is a null point) and $S_2 = S_3 := S$, then $q = q(a_2, a_3) = \frac{4(S-1)}{S^2}$.